The exam's answer

Question No. 1

(16 marks)

(a) Let A and B be events with P(A)=3/8, P(B)=5/8 and P(AUB)=3/4. **Find P(A/B) and P(B/A).**

The answer:

$$P(A\cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A\cap B) = P(A) + P(B) - P(A \cup B)$$
$$= \frac{3}{8} + \frac{5}{8} - \frac{3}{4} = \frac{1}{4}$$
$$P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{1}{4} / \frac{5}{8} = \frac{2}{5}$$
$$P(B/A) = \frac{P(A\cap B)}{P(A)} = \frac{1}{4} / \frac{3}{8} = \frac{2}{3}$$

(b) A box contains 7 red marbles and 3 white marbles. Three marbles are drawn from the box one after the other. **Find the probability that the first two are red and the third is white.**

The answer:

The Probability that the first marble is red = P(E1)=7/10
The Probability that the second marble is red = P(E2)=6/9
The Probability that the third marble is white = P(E3)=3/8

P(E) = P(E1).P(E2).P(E3)
=(7/10).(6/9).(3/8)=7/40 = 0.175

(c) In a certain collage, 25% of the students failed mathematics, 15% of the students failed chemistry and 10% of the students failed both. A student is selected at random:

i. If he failed chemistry, **what is the probability that he failed mathematics**?
ii. If he failed mathematics, **what is the probability that he failed chemistry?**
iii. What is the probability that he failed mathematics or chemistry?

The answer:

E1: Students failed in mathematics
E2: Students failed in chemistry
E3: Students failed in Both
P(E1)=0.25
P(E2)=0.15
P(E3)=P(E1∩E2)=0.1
i) P(E1|E2)=P(E1∩E2)/P(E2) =0.1/0.15=2/3
ii) P(E2|E1)=P(E2∩E1)/P(E1) =0.1/0.25=2/5
iii) P(E1UE2)=P(E1)+P(E2)-P(E1∩E2) =0.25+0.15-0.1=0.3

Question No. 2

(18 marks)

(a) Medical research has shown that a certain type of chemotherapy is successful 70% of the time when used to treat skin cancer. Suppose five skin cancer patients are treated with this type of chemotherapy and let x equal the number of successful cures out of the five. The probability distribution of x is given in the following table:

X	**0**	**1**	**2**	**3**	**4**	**5**
P(x)	**0.002**	**0.029**	**0.132**	**0.309**	**0.360**	**0.168**

a) Find $\mu = E(x)$. Interpret the result.

The answer:

$\mu = E(x) = \sum x\, p(x)$
$= 0*0.002 + 1*0.029 + 2*0.132 + 3*0.309 + 4*0.309 + 5*0.168$
$= 3.5$

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b) Find $\sigma = \sqrt{E(x-\mu)}^{2}$. Interpret the result.

The answer:

$\delta = \sqrt{E(x-\mu)^2}$
$\delta^2 = E(x-\mu)^2 = E(x)^2 - \mu^2$
$E(x^2) = \sum x^2 P(x)$
$= 0^2*0.002 + 1^2*0.029 + 2^2*0.132 + 3^2*0.309 + 4^2*0.360 + 5^2*0.168$
$= 13.298$
$\delta^2 = E(x)^2 - \mu^2 = 13.298 - (3.5)^2 = 13.298 - 12.25 = 1.05$
$\delta = 1.02$

(b) Prove that for any random variable x:

a) $E(ax + b) = a\,E(x) + b$

The answer:

$E(ax+b) = \int_{-\infty}^{\infty}(ax+b)p(x)dx = \int_{-\infty}^{\infty} ax\, p(x)dx + \int_{-\infty}^{\infty} b\, p(x)dx$
$= a\int_{-\infty}^{\infty} xp(x)dx + b\int_{-\infty}^{\infty} p(x)dx = aE(x)+b = R.H.S$

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b) $V(ax + b) = a^2\, V(x)$

The answer:

$V(ax+b) = E[\,(ax+b) - E(ax+b)]^2 = E[\,ax + b - aE(x) + b]^2 = E[\,ax - aE(x)]^2 = a^2 E[x-\mu]^2 = a^2\, V(x) = R.H.S$

(c) Let x be a continuous random variable with density:

$$f(x) = \begin{cases} K(2-x) & 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Evaluate K and find the cumulative distribution function.

The answer:

$\because F(x)$ *is a density function*

$$\therefore \int_{-\infty}^{\infty} F(x)\, dx = 1$$

$$\int_{0}^{2} K(2-x)\, dx = 1$$

$$K(2x - \frac{x^2}{2})\Big|_0^2 = 1$$

$$K\left(4 - \frac{4}{2}\right) = 1$$

$\therefore 2K = 1$

$$\therefore K = \frac{1}{2}$$

The cumulative distribution function:

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

$$\because K = \frac{1}{2}$$

Then

$$f(x) = \begin{cases} (2-x)/2 & ,0 \le x \le 2 \\ 1 & ,elsewhere \end{cases}$$

$F(x)=\int_{-\infty}^{x} f(x)dx$

$-\infty \le x \le 0$	$f(x)=0$	$F(x)=0$
$0 \le x \le 2$	$f(x) = \frac{2-x}{2}$	$F(x)= F(0)+\int_0^x \frac{2-x}{2} dx = x - \frac{x^2}{4}$
$2 \le x \le \infty$	$f(x)=0$	$F(x)= F(2)+0=1$

$$F(x) = \begin{cases} 0 & ,-\infty \le x \le 1 \\ \left(x - \frac{x^2}{4}\right) & ,0 \le x \le 2 \\ 1 & ,x \ge 2 \end{cases}$$

Question No. 3

(18 marks)

(a) A fair die is tossed. Let X denote twice the number appearing, and let Y denote 1 or 4 according as an odd or an even number appears. **Find the probability, expectation, variance and standard deviation of:**

i) X

The answer:

X is twice no appearing

$x|1=2$, $x|2=4$, $x|3=6$, $x|4=8$, $x|5=10$, $x|6=12$

distribution

x	2	4	6	8	10	12
P(x)	1/6	1/6	1/6	1/6	1/6	1/6

$\mu=2/6+4/6+6/6+8/6+10/6+12/6=7$

$E(x^2)=4/6+16/6+36/6+64/6+100/6+144/6=60.7$

$Var=E(x^2)-\mu=11.7$

S.D=3.4

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ii) Y

The answer:

$y|1=y|3=y|5=1$, $y|2=y|4=y|6=4$

$p(y=1)=p(\{1,3,5\})=1/2$

$p(y=4)=p(\{2,4,6\})=1/2$

y	1	4
P(y)	1/2	1/2

$\mu=1/2+4/2=2.5$

$E(y^2)=1/2+16/2=8.5$

$Var(y)=8.5-(2.5)^2=2.25$

S.D= 1.5

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iii) X+Y

The answer:

$x+y|1=3$, $x+y|2=8$, $x+y|3=7$, $x+y|4=12$, $x+y|5=11$, $x+y|6=16$

x+y	3	7	8	11	12	16
P(x+y)	1/6	1/6	1/6	1/6	1/6	1/6

$\mu=3/6+7/6+8/6+11/6+12/6+16/6=9.5$

$E(x+y)^2=9/6+49/6+64/6+122/6+144/6+256/6=107.166$

$Var(x+y)=107.166-(9.5)^2=16.912$

S.D= 4.11

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iv) XY

The answer:

$xy|1=2$, $xy|2=16$, $xy|3=6$, $xy|4=32$, $xy|5=10$, $xy|6=48$

xy	2	6	10	16	32	48
P(xy)	1/6	1/6	1/6	1/6	1/6	1/6

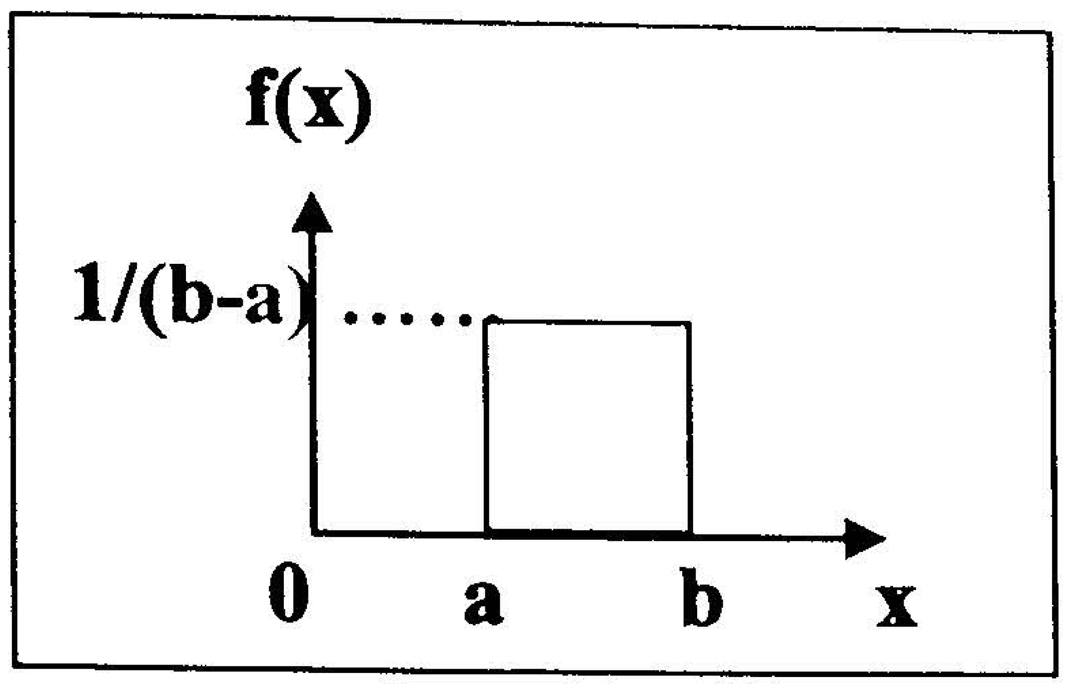
$E(xy)=2/6+6/6+10/6+16/6+32/6+48/6=19$

$E(xy^2)=4/6+36/6+100/6+256/6+1024/6+2304/6=620.66$

$Var(xy)=620.66-(19)^2=259.66$

S.D=16.114

(b) For the uniform distribution shown in the following figure,



prove that:

a) Mean = (b+a)/2

<u>The answer:</u>

$$Mean = \int_{-\infty}^{\infty} x\,.p(x).dx$$

$$= \int_a^b \frac{1}{b-a}.x.dx = \frac{1}{b-a}.\frac{x^2}{2}$$

$$= \frac{1}{2(b-a)}.[b^2 - a^2] = \frac{(b-a).(b+a)}{2.(b-a)} = \frac{b+a}{2}$$

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b) Variance = (b-a)²/12

<u>The answer:</u>

$$E(x^2) = \int_a^b \left(\frac{1}{b-a}\right).x^2.dx = \frac{1}{b-a}.\frac{x^3}{3} = \frac{1}{3.(b-a)}.(b^3 - a^3)$$

$$= \frac{1}{3.(b-a)}.(b-a).(b^2 + ab + a^2)$$

$$var = \frac{(b^2 + ab + a^2)}{3} - \frac{(b+a)^2}{4}$$

$$= \frac{4(b^2 + ab + a^2)}{12} - \frac{3(b^2 + 2ab + a^2)}{12} = \frac{(b^2 - 2ab + a^2)}{12} = \frac{(b-a)^2}{12}$$

(c) A family has 6 children. Find the probability P that there are:

i. 3 boys and 3 girls.

<u>The answer:</u>

$$P(x) = \binom{n}{x} (p)^x\,(1-p)^{n-x}$$

$$P(6,3,1/2) = \binom{6}{3} (1/2)^3\,(1-1/2)^{6-3}$$

P(6,3,1/2)=(20)*(1/8)*(1/8) = 5/16 = 0.3125

ii. Fewer boys than girls.

The answer:

P(0 boys)+p(1boy)+p(2boys)=

$$(1/2)6 + \binom{6}{1}(1/2)5 + \binom{6}{2}(1/2)2\ (1/2)4 = 11/32 = 0.3437$$

Question No. 4

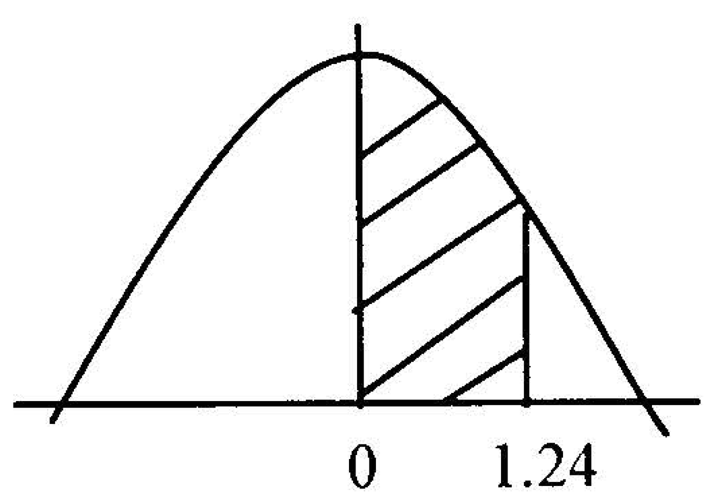
(18 marks)

(a) Let X be a random variable with the standard normal distribution Φ. **Find:**

i. $P(0 \le X \le 1.24)$

The answer:

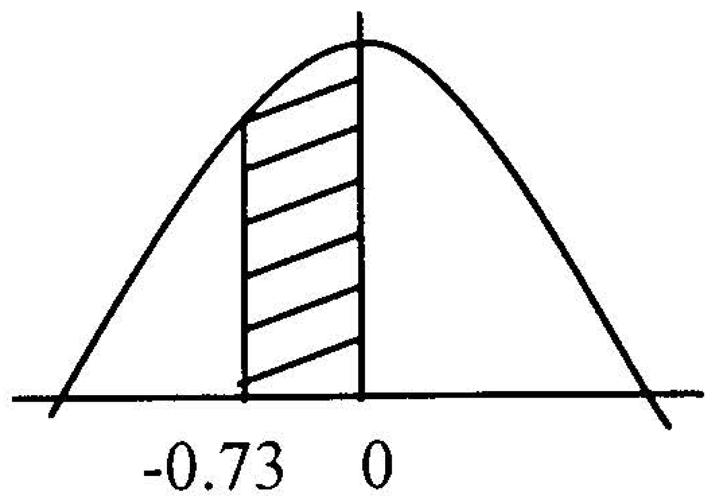
$P(0 \le X \le 1.24)$ is equal to the area under the standard normal curve between 0 and 1.24. by using the attached table $P(0 \le X \le 1.24) = 0.3925$



ii. $P(-0.73 \le X \le 0)$

The answer:

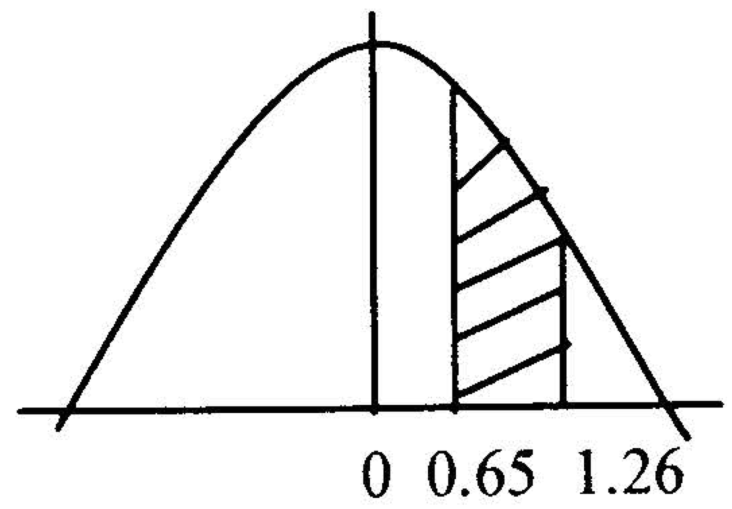
$P(-0.73 \le X \le 0) = P(0 \le X \le 0.73) = 0.2673$



i. $P(0.65 \le X \le 1.26)$

The answer:

$P(0.65 \le X \le 1.26) = P(0 \le X \le 1.26) - P(0 \le X \le 0.65)$
$= 0.3962 - 0.2422 = 0.1540$



(b) The mean and standard deviation on an examination are 74 and 12 respectively. **Find the scores in standard units of students receiving marks**:

i) 65

<u>The answer:</u>

$$\mathbf{t = (x-\mu)/\sigma = (65-74)/12 = -0.75}$$

ii) 74

<u>The answer:</u>

$$\mathbf{t = (x-\mu)/\sigma = (74-74)/12 = 0}$$

iii) 86

<u>The answer:</u>

$$\mathbf{t = (x-\mu)/\sigma = (86-74)/12 = 1}$$

iv) 92

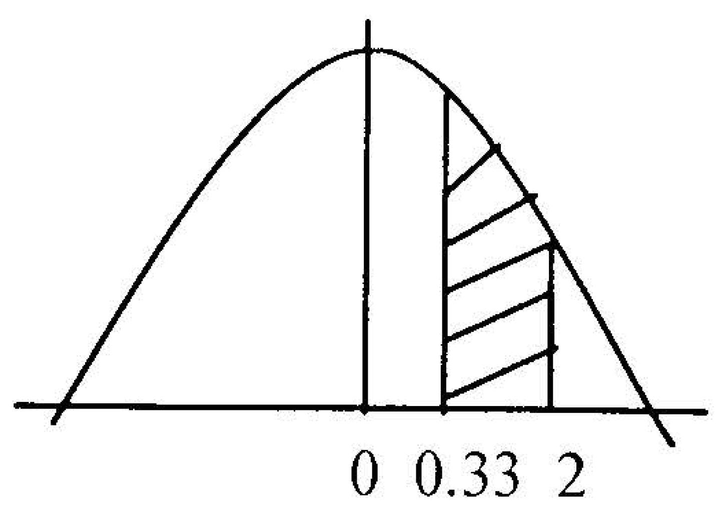
<u>The answer:</u>

$$\mathbf{t = (x-\mu)/\sigma = (92-74)/12 = 1.5}$$

(c) Suppose the temperature during June is normally distributed with mean $20^{o}C$ and standard deviation 3.33 deg. Find the **probability** P that the temprature is between $21.11^{o}C$ and $26.66^{o}C$.

<u>The answer:</u>

$21.11^{o}C$ in standard units $= (21.11 - 20)/3.33 = 0.33$
$26.66^{o}C$ in standard units $= (26.66 - 20)/3.33 = 2$



Then

$P = P(26.66 \leq T \leq 21.11)$
$= P(0 \leq T^{*} \leq 2) - P(0 \leq T^{*} \leq 0.33)$
$= 0.4772 - 0.1293 = 0.3479$

Best wishes